

A ladder ellipse problem

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Abstract

We consider a problem similar to the well-known ladder box problem, but where the box is replaced by an ellipse. A ladder of a given length, s , with ends on the positive x and y axes, is known to touch an ellipse that lies in the first quadrant and is tangent to the positive x and y axes. We then want to find the height of the top of the ladder above the floor. We show that there is a value, $s = s_0$, such that there is only one possible position of the ladder, while if $s > s_0$, then there are two different possible positions of the ladder. Our solution involves solving an equation which is equivalent to solving a 4th degree polynomial equation.

1 Introduction

The well-known ladder box problem(see [1], [4]) involves a ladder of a given length, say s meters, with ends on the positive x and y axes, which touches a given rectangular box(a square in [5]) at its upper right corner. One then wants to determine how high the top of the ladder is above the floor. Other versions of problem([5]) ask how much of the ladder is between the wall(or floor) and the point of contact of the ladder with the box. We ask similar questions in this note, but where the box is replaced by an *ellipse*, E_0 , that lies in the first quadrant and is tangent to the positive x and y axes at the points $(c, 0)$ and $(0, d)$. For example, consider the ellipse with equation $x^2 + 4y^2 + 2xy - 8x - 16y + 16 = 0$, which is tangent to the positive x and y axes at the points $(4, 0)$ and $(0, 2)$. If the ladder has length 10 meters, then how high is the top of the ladder above the floor and how many positions of the ladder are possible ? One main difference here is that we now allow the ladder to be tangent at *any* point of E_0 rather than just at the upper right corner of a rectangular box. We suppose that the ladder touches the positive x and y axes at the points $(u, 0)$ and $(0, v)$, respectively, and we call such a ladder admissible. We then want to find u , which is the

height of the top of the ladder above the floor. It is not hard to show that the equation of E_0 must have the form

$$d^2x^2 + c^2y^2 + 2Cxy - 2cd^2x - 2c^2dy + c^2d^2 = 0, \quad (1)$$

and that if the equation of E_0 is given by (1), then E_0 is tangent to the positive x and y axes at the points $(c, 0)$ and $(0, d)$. Note that for (1) to be the equation of an ellipse, we need $c^2d^2 - C^2 > 0$, which is equivalent to

$$cd > |C|. \quad (2)$$

We now assume throughout that T is the triangle with vertices $(0, 0)$, $(u, 0)$, and $(0, v)$ with $u, v > 0$.

Remark: There is another way to look at this problem: Given an ellipse, E_0 , inscribed in a right triangle, T , suppose that we know the length of the hypotenuse of T and the points of tangency of E_0 with the other two sides of T . We want to find the lengths of the other sides of T .

It is useful now to derive another form for the equation of E_0 which depends on two parameters, which we denote by w and t . The following lemma and proposition were proven in [3] for the case when T is the unit triangle. Throughout we let I denote the open interval $(0, 1)$ and I^2 the unit square $= (0, 1) \times (0, 1)$.

Lemma 1 *Let V be the interior of the medial triangle of $T = \text{triangle with vertices at the midpoints of the sides of } T$. Then $V = \{(x_{w,t}, y_{w,t})\}_{(w,t) \in I^2}$, where*

$$x_{w,t} = \frac{1}{2} \frac{tu}{w + (1-w)t} \text{ and } y_{w,t} = \frac{1}{2} \frac{wv}{w + (1-w)t}.$$

Proof. It follows easily that $(x, y) \in V$ if and only if

$$\begin{aligned} \frac{v}{2} - \frac{v}{u}x &< y < \frac{v}{2} \\ 0 &< x < \frac{u}{2}. \end{aligned} \quad (3)$$

Now suppose that $(w, t) \in I^2$. We want to show that $(x_{w,t}, y_{w,t}) \in V$; First, $x_{w,t} - \frac{u}{2} = -\frac{1}{2} \frac{uw(1-t)}{w + (1-w)t} < 0$ and $y_{w,t} - \frac{v}{2} = -\frac{1}{2} \frac{v(1-w)t}{w + (1-w)t} < 0$, which implies that $0 < x_{w,t} < \frac{u}{2}$ and $y_{w,t} < \frac{v}{2}$; Also, $\frac{v}{u}x_{w,t} + y_{w,t} - \frac{v}{2} = \frac{1}{2} \frac{vwt}{w + (1-w)t} > 0$, and so $\frac{v}{2} - \frac{v}{u}x_{w,t} < y_{w,t}$, which implies that $(x_{w,t}, y_{w,t}) \in V$ by (3). Conversely, suppose that $(x_{w,t}, y_{w,t}) \in V$. We want to show that $(w, t) \in I^2$; Solving the system of equations

$$\begin{aligned} \frac{1}{2} \frac{tu}{w + (1-w)t} &= x \\ \frac{1}{2} \frac{wv}{w + (1-w)t} &= y \end{aligned}$$

yields the unique solution $w = w_{x,y} = \frac{2uy + 2vx - uv}{2xv}$, $t = t_{x,y} = \frac{2uy + 2vx - uv}{2uy}$;

Substituting $x = x_{w,t}$, $y = y_{w,t}$ and using (3) easily implies that

$$0 < w_{x,y}, t_{x,y} < 1 \text{ and so } (w_{x,y}, t_{x,y}) \in I^2. \blacksquare$$

Proposition 1 (i) Suppose that E_0 is an ellipse inscribed in T . Then the equation of E_0 is given by

$$(vw)^2x^2 + (ut)^2y^2 + 2wt(2w + 2t - 2wt - 1)uvxy - 2ut(vw)^2x - 2wv(ut)^2y + (uvwt)^2 = 0 \quad (4)$$

for some $(w, t) \in I^2$.

(ii) If E_0 is an ellipse with equation (4) for some $0 < t < 1$, $0 < w < 1$, then E_0 is tangent to the three sides of T at the points $T_1 = (ut, 0)$, $T_2 = (0, vw)$, and $T_3 = \left(\frac{ut(1-w)}{w+t-2wt}, \frac{vw(1-t)}{w+t-2wt}\right)$.

Proof. Note that the denominator in both coordinates of T_3 is nonzero since $w + t - 2wt > 0$ holds for any $(w, t) \in I^2$ by the Arithmetic–Geometric Mean inequality. First, suppose that E_0 is given by (4) for some $(w, t) \in I^2$. Then E_0 has the form $Ax^2 + By^2 + 2Cxy + Dx + Ey + F = 0$, where $AB - C^2$ easily simplifies to $4(uvwt)^2(1-w)(1-t)((1-t)w+t) > 0$, and thus (4) defines the equation of an ellipse. Also, $AE^2 + BD^2 + 4FC^2 - 2CDE - 4ABF = 16(uvwt)^4(1-w)^2(1-t)^2 > 0$, which implies that such an ellipse is non-trivial. Now let $H(x, y)$ denote the left hand side of (4). Since $H(T_1) = H(T_2) = H(T_3) = 0$, the three points T_1, T_2 , and T_3 lie on E_0 . Differentiating both sides of the equation in (4) with respect to x yields $\frac{dy}{dx} = D(x, y)$, where $D(x, y) = -\frac{vw - 2uy(1-w)t^2 + u(vw - 2wy + y)t - vwx}{ut(2w^2vx - 2vwx - uy + vwu)t - vwx(2w - 1)}$; $D(T_1) = 0 = \text{slope of horizontal side of } T$ and $D(T_3) = -\frac{v}{u} = \text{slope of the hypotenuse of } T$; When $x = 0, y = w$, the denominator of $D(x, y)$ equals 0, but the numerator of $D(x, y)$ equals $2uvwt(1-t)(1-w) \neq 0$. Thus E_0 is tangent to the vertical side of T . For any simple closed convex curve, such as an ellipse, tangent to each side of T then implies that that curve lies in T . Since it follows easily that T_1, T_2 , and T_3 lie on the three sides of T , that proves that E_0 is inscribed in T . Second, suppose that E_0 is an ellipse inscribed in T . It is well known [2] that each point of V , the medial triangle of T , is the center of one and only one ellipse inscribed in T , and thus the center of E_0 lies in V . By Lemma 1, the center of E_0 has the form $(x_{w,t}, y_{w,t})$, $(w, t) \in S$. Now it is not hard to show that each ellipse given by (4) also has center $(x_{w,t}, y_{w,t})$. Since we have just shown that (4) represents a family of ellipses inscribed in T as (w, t) varies over I^2 , if E_0 were not given by (4) for some $(w, t) \in I^2$, then there would be two ellipses inscribed in T and with the same center. That cannot happen since each point of V is the center of only one ellipse inscribed in T . That proves (i). We have also just shown that if E_0 is given by (4), then E_0 is tangent to the three sides of T at the points T_1, T_2 , and T_3 , which proves (ii). \blacksquare

Now if we know that E_0 lies in the first quadrant and is tangent to the positive x and y axes at the points $(c, 0)$ and $(0, d)$, then E_0 is inscribed in some triangle, T , with vertices $(0, 0)$, $(u, 0)$, and $(0, v)$ with $u, v > 0$. By Proposition 1(ii), $c = ut$ and $d = vw$; Substituting into (4) yields

$$d^2x^2 + c^2y^2 + 2cd(2w + 2t - 2wt - 1)xy - 2cd^2x - 2c^2dy + c^2d^2 = 0. \quad (5)$$

Comparing (1) and (5) yields $C = cd(2w + 2t - 2wt - 1)$, which implies that

$$w + t - wt = J, \quad J = \frac{1}{2} \left(1 + \frac{C}{cd} \right). \quad (6)$$

Note that by (2) $cd > C$ and $cd > -C$, which implies that $0 < J < 1$; We want to choose (w, t) so that the ladder has the given length, s ; Using $u = \frac{c}{t}, v = \frac{d}{w}$, we have $s^2 = u^2 + v^2 = \frac{c^2}{t^2} + \frac{d^2}{w^2}$, and since $w = \frac{J-t}{1-t}$ from (6) we have $s^2 = f(t)$, where

$$f(t) = \frac{c^2}{t^2} + \frac{d^2(1-t)^2}{(J-t)^2}. \quad (7)$$

If $t \in I$, then $\frac{J-t}{1-t} > 0$ if and only if $t < J$; Also, if $t < 1$, then $1 - \frac{J-t}{1-t} = \frac{1-J}{1-t} > 0$; Thus we have

$$w = \frac{J-t}{1-t} \in I \iff t < J, \text{ where } t, J \in I. \quad (8)$$

Thus for given s , using (7), we want to solve the equation $f(t) = s^2$ for $t \in (0, J)$; For example, for the ellipse with equation $x^2 + 4y^2 + 2xy - 8x - 16y + 16 = 0$, multiplying thru by 4 yields the form of the equation given in (1), with $c = 4$, $d = 2$, and $C = 4$; Suppose, say that $s = 10$. That gives $f(t) = \frac{16}{t^2} + \frac{4(1-t)^2}{(3/4-t)^2}$ and it is not hard to show that the equation $f(t) = 100$ has two solutions $t_1 = \frac{2}{3}$ and $t_2 \approx 0.43$ in $(0, J)$, $J = \frac{3}{4}$; The corresponding w values are then $w_1 = \frac{J-t_1}{1-t_1} = \frac{1}{4}$ and $w_2 = \frac{J-t_2}{1-t_2} \approx 0.56$, which gives $u_1 = \frac{c}{t_1} = 6$, $v_1 = \frac{d}{w_1} = 8$, $u_2 = \frac{c}{t_2} \approx 9.35$, and $v_2 = \frac{d}{w_2} \approx 3.57$. The corresponding points where the ladder is tangent to E_0 are $T_{3,1} = \left(\frac{36}{7}, \frac{8}{7} \right)$ and $T_{3,2} \approx (3.48, 2.24)$; For this example there are *two* different positions of the ladder, which is analagous to what happens with the ladder box problem. But are there always two different positions of the ladder ? To help answer this, first we assume that there is an admissible ladder of length s which touches E_0 , so it follows that the equation

$f(t) = s^2$ has at least one solution in $(0, J)$; Since $\lim_{t \rightarrow 0^+} f(t) = \lim_{t \rightarrow J^-} f(t) = \infty$, $f(t) = s^2$ must have at least two solutions in $(0, J)$, counting multiplicities. Now $f'(t) = -2 \left(\frac{c^2}{t^3} - \frac{d^2(1-t)(1-J)}{(J-t)^3} \right)$ and the function of $t, y = \frac{c^2}{t^3}$, is clearly decreasing on $(0, J)$; Since $\frac{d}{dt} \left(\frac{1-t}{(J-t)^3} \right) = \frac{(J-t)^2(3-J-2t)}{(J-t)^6} > 0$ on $(0, J)$, the function of $t, y = \frac{d^2(1-t)(1-J)}{(J-t)^3}$ is increasing on $(0, J)$; Thus the equation $\frac{c^2}{t^3} = \frac{d^2(1-t)(1-J)}{(J-t)^3}$ has at most one solution in $(0, J)$; Since $\lim_{t \rightarrow 0^+} f'(t) = -\infty$ and $\lim_{t \rightarrow J^-} f'(t) = \infty$, f' has at least one root in $(0, J)$; Hence f' has exactly one root, say t_0 , in $(0, J)$, and $\begin{cases} f'(t) < 0 & \text{if } 0 < t < t_0 \\ f'(t) > 0 & \text{if } t_0 < t < J \end{cases}$; That in turn implies that f is decreasing on $(0, t_0)$ and is increasing on (t_0, J) and so $f(t) = s^2$ has most two solutions in $(0, J)$; So we can conclude that $f(t) = s^2$ has exactly two solutions in $(0, J)$, *counting multiplicities*. The only way that there would be only one position of the ladder is if $f(t) - s^2$ has a double root in $(0, J)$; Can this actually happen? To help answer this question, let E_R = rightmost open arc of E_0 between the points, P_H and P_V , on E_0 where the tangents are horizontal or vertical. Clearly there is an admissible ladder tangent to E_0 at any point of E_R . As the point of tangency approaches P_H or P_V , s approaches ∞ . Hence there is a unique value $s_0 > 0$ such that there is an admissible ladder of length s tangent to E_0 at any point of E_R if and only if $s \geq s_0$. How does one find s_0 ? $s_0 = f(t_0)$, where t_0 is the unique root of f' in $(0, J)$ discussed above. For $s = s_0$, there is only one position of the ladder, while if $s > s_0$, then there are two different positions of the ladder. For the example above, $f'(t)$ has one root in $(0, J)$, $t_0 \approx 0.58$; Then $s_0 = f(t_0) \approx 72$.

Remark: Solving $f(t) = s^2$ is equivalent to solving the 4th degree polynomial equation

$$p_s(t) = 0, p_s(t) = (c^2 - s^2 t^2)(J - t)^2 + d^2 t^2 (1 - t)^2. \quad (9)$$

Note that one approach for solving the ladder box problem also involves solving a 4th degree polynomial equation.

Remark: Another way to solve this problem would be to use an affine map to send E_0 to a circle, C , inscribed in a triangle, T , which is now not necessarily a right triangle. Then the problem becomes: Suppose that we know the length of a side, c , of a triangle, T' , and we know that a circle, C , is inscribed in T' and we know the points of tangency of the other two sides, a and b ; Can one find the lengths of a and b , and if yes, is the answer unique?

Special Case: Not surprisingly, things simplify somewhat when the ellipse, E_0 , is a **circle**. In that case $d = c$, $C = 0$, and $J = \frac{1}{2}$. The polynomial $p_s(t)$ from (9) factors as a product of two quadratics:

$$p_s(t) = -\frac{1}{4}(2(s - c)t^2 - (s - 2c)t - c)(2(s + c)t^2 - (s + 2c)t + c); \text{ It is then}$$

easy to show that the critical number s_0 of f is given by $2(\sqrt{2} + 1)c$, so that there are two different positions of the ladder when $s > 2(\sqrt{2} + 1)c$.

References

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